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RESEARCH ARTICLE

Undecidability of the word problem for Yamamura's HNN-extension under nice conditions

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Abstract The solvability of the word problem for Yamamura's HNN-extensions $[S; A_1, A_2; \varphi]$ has been proved in some particular cases. However, we show that, contrary to the group case, the word problem for $[S; A_1 A_2; \varphi]$ is undecidable even if we consider S to have finite $\mathscr{R}$-classes, A_1 and A_2 to be free inverse subsemigroups of finite rank and with zero, and φ, φ^{-1} to be computable.

Keywords HNN-extension · Inverse Semigroups · Word Problem · Undecidability · Amalgams of Inverse Semigroups

1 Introduction

The concept of HNN-extension was originally introduced for groups by Higman, Neumann and Neumann, who showed that if A_1 and A_2 are isomorphic subgroups of a group S, then it is possible to find a group H containing S such that A_1 and A_2 are conjugate to each other in H and such that S is embeddable in H. This concept may be generalized to the class of semigroups in different ways, see for instance

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[1,4,6,7,9]. In the class of inverse semigroups there are two main approaches, one given by Gilbert [5] and the other one by Yamamura [19]. For basic definitions on inverse semigroups we refer the reader to standard books on the subject [8,12,15]. In this paper we consider HNN-extension in the sense of Yamamura, and we refer to [21] to clarify the connections with Gilbert's case. The issue that it is considered here has an algorithmic nature: the word problem in Yamamura's HNN-extension $[S; A_1, A_2; \varphi]$ under mild conditions on the tuple S, A_1, A_2, φ. In some cases this problem has been proved to be decidable, for instance when S is finite [16]. Another case for which the word problem is decidable is in the lower bounded case [10] under the following conditions:

- S has solvable word problem;
- A_1 and A_2 have solvable membership problem and φ is computable;
- there is an algorithm for deciding whether or not the sets $U_{A_i}(e) = \{u \in A_i : u \geq e\}$, $i = 1, 2$, are empty for every idempotent e of S;
- there is an algorithm for computing the minimum idempotents of $U_{A_i}(e)$, $i = 1, 2$ (in case it is non-empty);
- there is an algorithm to decide whether a path connecting two given vertices of the Schützenberger automaton relative to $[S; A_1, A_2; \varphi]$ is labelled by an element from A_1 or A_2.

Another case in which the word problem is decidable is when S is a free inverse semigroup and A_1 and A_2 are finitely generated subsemigroups [10]. All these results are shown using graphical methods, introduced by Stephen in [18], to obtain "graphical normal forms". For instance, Munn trees are "graphical normal forms" for free inverse semigroups [14]. Britton's Lemma for HNN-extension of groups provides a normal form in the usual sense. This normal form is effectively computable when the original group S has solvable word problem, the subgroups A_1 and A_2 have solvable membership problem, and φ, φ^{-1} are computable (see Corollary 2.2 of [13]). As a byproduct, the word problem for HNN-extensions of groups with the aforementioned properties is decidable. In the inverse semigroup case there is no analogue of Britton's Lemma, and the use of graphical methods based on Schützenberger automata is the key for proving the decidability of the word problem in the previously mentioned cases. In [20] the author generalizes Britton's Lemma for the case of a locally full HNN-extension of an inverse semigroup. This fact provides a normal form that yields the decidability of the word problem under analogous conditions to the group case mentioned above [20, Theorem 3.6]. However, it may be the case that, in general, such a normal form does not exist. This paper partially supports this point of view, and it offers a result on the opposite direction with respect to the previous decidability results. Indeed, we show that even assuming nice conditions on the tuple S, A_1, A_2, φ, the word problem for Yamamura's HNN-extensions of inverse semigroups may be undecidable. In particular, we show that the following main theorem holds.

Theorem 1 *The word problem for Yamamura's HNN-extensions of inverse semigroups* $[S; A_1, A_2; \varphi]$ *is undecidable even if we assume the following conditions:*

- *S has finite $\mathscr{R}$-classes (therefore solvable word problem);*
- *the membership problem for A_1, A_2 in S is decidable, and $A_1 \simeq A_2$ is a free inverse semigroup with zero and finite rank;*

– *φ and its inverse are computable functions.*

It is noteworthy that, although inverse semigroups may be seen to be very close to groups (by the Vagner-Preston Theorem inverse semigroups may be represented as partial one-to-one maps, while groups as bijective maps) the result obtained here contrasts with the groups case.

The proof of Theorem 1 relies on the following undecidability result regarding the word problem under similar conditions for amalgams of inverse semigroups:

Theorem 2 [17, Theorem 1] *The word problem for an amalgam $[S_1, S_2; U, \omega_1, \omega_2]$ of inverse semigroups may be undecidable even if we assume the following conditions:*

- *S_1, S_2 have finite $\mathscr{R}$-classes;*
- *U is a free inverse semigroup with zero of finite rank;*
- *the membership problem of $\omega_i(U)$ is decidable in S_i for $i = 1, 2$;*
- *ω_1, ω_2 and their inverses are computable functions;*

Roughly speaking we show that, given an amalgam $[S_1, S_2; U, \omega_1, \omega_2]$ with the aforementioned properties, we may effectively build an HNN-extension S^* with the conditions stated in Theorem 1 such that if the word problem in S^* is decidable, then it is also decidable for $[S_1, S_2; U, \omega_1, \omega_2]$. This leads to a contradiction in view of Theorem 2. The paper is organized as follows. In Sect. 2 we recall some basic notions and terminology regarding Schützenberger automata of Yamamura's HNN-extensions. We also recall a relationship between Yamamura's HNN-extensions and amalgams of inverse semigroups, investigated in [3], that is crucial in the aforementioned reduction to the word problem in amalgams of inverse semigroups. Finally, in Sect. 3 we prove Theorem 1.

2 Basic Notions

Let X be a finite set and denote by X^{-1} the set $\{x^{-1} : x \in X\}$ of formal inverses of the elements of X where $^{-1}$ is the usual involution on $X \cup X^{-1}$. We put $S = Inv\langle X|R\rangle$ and we say that $\langle X|R\rangle$ is a *presentation* of the inverse semigroup S whenever $S = (X \cup X^{-1})^+/\theta$, where θ is the least congruence containing R and the Vagner's relation:

$$\nu = \{zz^{-1}z = z,\ zz^{-1}yy^{-1} = yy^{-1}zz^{-1},\ \forall z, y \in (X \cup X^{-1})^+\}.$$

We will assume throughout the paper that the inverse semigroups are finitely presented, i.e., X, R are finite. The following definition of HNN-extension is due to A. Yamamura [19].

Definition 1 Let S be an inverse semigroup $S = Inv\langle X|R\rangle$ and let $\varphi : A_1 \to A_2$ be an isomorphism of an inverse subsemigroup A_1 of S onto an inverse subsemigroup A_2 of S where $e \in A_1 \subseteq eSe$ and $f \in A_2 \subseteq fSf$ (or $e \notin A_1 \subseteq eSe$ and $f \notin A_2 \subseteq fSf$ for some idempotents $e, f \in S$). The inverse semigroup

$$S^* = Inv\langle S, t \mid t^{-1}at = \varphi(a),\ t^{-1}t = f,\ tt^{-1} = e,\ \forall a \in A_1\rangle$$

is called the *HNN-extension* of S associated with $\varphi : A_1 \to A_2$ and it is denoted by $[S; A_1, A_2; \varphi]$.

In the sequel we put $S^* = Inv\langle \overline{X} \mid R \cup R_{HNN}\rangle$ where $\overline{X} = X \cup \{t\}$ and we compactly denote by R_{HNN} the relations $t^{-1}at = \varphi(a)$, $t^{-1}t = f$, $tt^{-1} = e$, for every $a \in A_1$. We refer to the presentation $\langle \overline{X} \mid R \cup R_{HNN}\rangle$ as the standard presentation of S^*. Since we follow a combinatorial approach, we give some related notions and terminology regarding Schützenberger automata. We recall that the Schützenberger automaton $\mathscr{A}(\overline{X}, R \cup R_{HNN}; w)$ of a word $w \in (\overline{X} \cup \overline{X}^{-1})^+$ is the Cayley graph with respect to the standard presentation $\langle \overline{X} \mid R \cup R_{HNN}\rangle$ with initial state the element represented by ww^{-1} and final state the element represented by w. For further details and general properties of Schützenberger automata we refer the reader to Stephen's paper [18], and to the paper [16] for some basic terminology and notions of a Schützenberger automaton $\mathscr{A}(\overline{X}, R \cup R_{HNN}; w)$ from which this paper takes inspiration. Let $\Gamma = (V(\Gamma), E, \overline{X})$ be a deterministic inverse word graph on $\overline{X}$, where $V(\Gamma)$ is the set of vertices, $E \subseteq V(\Gamma) \times (\overline{X} \cup \overline{X}^{-1}) \times V(\Gamma)$ is the set of edges with the property that if $(v, a, v') \in E$, $a \in \overline{X} \cup \overline{X}^{-1}$, then $(v', a^{-1}, v) \in E$. The condition to be deterministic is expressed by the property that if $(v, a, p), (v, a, p') \in E$, $a \in (\overline{X} \cup \overline{X}^{-1})$, then $p = p'$. An inverse word automaton (for short an automaton) is a pair (v, Γ, v') where $v, v' \in V(\Gamma)$ are the initial and final states, respectively. A subgraph Δ of Γ with at least an edge is called *S-lobe* if it is a maximal connected inverse subgraph on X. A *t-edge* is an edge of Γ labelled by t. Two vertices v_1, v_2 are called *t-adjacent* if they are connected by a t-edge, i.e., if either (v_1, t, v_2) or (v_2, t, v_1) are edges of Γ. Two distinct S-lobes Δ_1, Δ_2 are called *adjacent* if there are two t-adjacent vertices $v_1 \in V(\Delta_1)$ and $v_2 \in V(\Delta_2)$. The *lobe graph* of the inverse word graph Γ is the directed graph $G(\Gamma)$ whose vertices are the S-lobes of Γ and there is a directed edge (Δ_1, Δ_2) from an S-lobe Δ_1 to an S-lobe Δ_2 if (v_1, t, v_2) is a t-edge with $v_1 \in \Delta_1$, $v_2 \in \Delta_2$. In [16], where it is considered the case in which the original semigroup S is finite, it is shown that $\mathscr{A}(\overline{X}, R \cup R_{HNN}; w)$ has a particular shape called t-opuntoid automaton (or t-opuntoid graph in case we just consider the underlying graph). Since here we are dealing with the general case, we are just interested in a weaker version, that we call *weak t-opuntoid* graphs defined as follows:

Definition 2 An inverse word graph (automaton) on $\overline{X}$ is called *weak t-opuntoid* if it is deterministic and its lobe graph is a tree.

By [10, Theorem 2.2.1] we conclude that every Schützenberger automaton with respect to the standard presentation of S^* is a weak t-opuntoid automaton.

As we have already mentioned in the introduction, the proof of Theorem 1 relies on an analogous result for amalgams of inverse semigroups. We now recall some basic notions and terminology concerning amalgams of inverse semigroups.

Definition 3 Let S_1, S_2 be two inverse semigroups with $S_1 \cap S_2 = \emptyset$, and let $\omega_i : U \hookrightarrow S_i$, $i \in \{1, 2\}$ be two monomorphisms. The tuple $[S_1, S_2; U, \omega_1, \omega_2]$ is called the *amalgam of* S_1 *and* S_2 *with core* U. The free product of S_1 and S_2 amalgamating U is the inverse semigroup $S = S_1 *_U S_2$ such that:

(i) there is a homomorphism $\sigma_i : S_i \to S$ for every $i \in \{1, 2\}$ and $\sigma_1 \circ \omega_1 = \sigma_2 \circ \omega_2$,
(ii) for every inverse semigroup T and for every pair $\{\phi_i : S_i \to T, i = 1, 2\}$ of homomorphisms such that $\omega_1 \circ \phi_1 = \omega_2 \circ \phi_2$, there is a unique homomorphism $\phi : S \to T$ such that $\sigma_i \circ \phi = \phi_i$ for each $i \in \{1, 2\}$.

Notice that the free product and the amalgamated free product of inverse semigroups are the coproduct and the pushout in the category of inverse semigroups, respectively. In the sequel we assume the semigroups S_1, S_2 to be finitely presented with presentations $\langle X_1|R_1\rangle$, $\langle X_2|R_2\rangle$, respectively, with $X_1 \cap X_2 = \emptyset$. Then, $S_1 *_U S_2$ is presented by $\langle X|R \cup R_W\rangle$ with $X = X_1 \cup X_2$, $R = R_1 \cup R_2$, $R_W = \{\omega_1(u) = \omega_2(u) : u \in U\}$ where, with a slight abuse of notation, for $i \in \{1, 2\}$ the symbol $\omega_i(u)$ denotes a word $w_i \in (X_i \cup X_i^{-1})^+$ representing the element $\omega_i(u)$. This presentation is called the standard presentation of $S_1 *_U S_2$.

Let Γ be an inverse word graph (automaton) on X, a *lobe* of Γ is a maximal connected inverse subgraph whose edges are labelled by elements of $(X_1 \cup X_1^{-1})$ (a lobe colored by 1) or by $(X_2 \cup X_2^{-1})$ (a lobe colored by 2). Two lobes Δ_1, Δ_2 are said to be adjacent if they share at least one common vertex, i.e., $V(\Delta_1) \cap V(\Delta_2) \neq \emptyset$. In this context, the *lobe graph* of Γ is the undirected graph whose vertices are lobes of Γ and whose edges correspond to the adjacency of lobes. In [2] it is shown that the Schützenberger automata of words with respect to the standard presentation of the free product with amalgamation $S_1 *_U S_2$ of two finite inverse semigroups S_1, S_2 are automata with a particular shape called opuntoid. Similarly to what we did before, we define a weaker notion of opuntoid. A *weak opuntoid* graph (automaton) is a deterministic inverse word graph (automaton) on X whose lobe graph is a tree. By [3, Theorem 3.6] we deduce that Schützenberger graphs (automata) relative to the standard representation of $S_1 *_U S_2$ are weak opuntoid graphs (automata).

We may associate to each amalgam $[S_1; S_2, U; \omega_1, \omega_2]$ a particular Yamamura's HNN-extension. We briefly recall this construction, and we refer the reader to [3] for further details. For each $i \in \{1, 2\}$ consider an element $e_i \notin X_i$ and let $S_i^{e_i}$ be the semigroup with adjoined identity e_i. Note that $S_i^{e_i}$ is presented by $\langle X_i \cup \{e_i\}|R_i^1\rangle$, where

$$R_i^1 = R_i \cup \{e_i^2 = e_i,\ e_i x_i = x_i e_i = x_i : \forall x_i \in X_i\}.$$

Let U^1 be the inverse semigroup obtained by adjoining the identity 1 to U. The embedding $\omega_i = U \hookrightarrow S_i$ may be extended naturally to an embedding $\omega_i^1 = U^1 \hookrightarrow S_i^{e_i}$ by putting $\omega_i^1(1) = e_i$ and $\omega_i^1(u) = \omega_i(u)$ for all $u \in U$. In this way we may consider the amalgam $[S_1^{e_1}, S_2^{e_2}; U^1, \omega_1^1, \omega_2^1]$. Note that $U_1^{e_1}, U_2^{e_2}$ embeds into the free product $S_1^{e_1} * S_2^{e_2}$, and with a slight abuse of notation we identify these subsemigroups with $U_1^{e_1}, U_2^{e_2}$ so that $(\omega_1^1)^{-1} \circ \omega_2^1$ is the isomorphism between them. Therefore, the Yamamura's HNN-extension associated to $[S_1^{e_1}; S_2^{e_2}, U^1; \omega_1^1, \omega_2^1]$ is given by the following HNN-extension:

$$[S_1^{e_1} * S_2^{e_2}; U_1^{e_1}, U_2^{e_2}; (\omega_1^1)^{-1} \circ \omega_2^1]$$

which is presented by $\langle \overline{X} | \overline{R} \cup R_{HNN} \rangle$, where $\overline{X} = X_1 \cup X_2 \cup \{e_1, e_2, t\}$, $\overline{R} = R_1^1 \cup R_2^1$, and

$$R_{HNN} = \{(\omega_1^1)^{-1} \circ \omega_2^1(u_1) = t^{-1} u_1 t : \ \forall u_1 \in U\} \cup \{tt^{-1} = e_1, \ t^{-1} t = e_2\}.$$

Let $S^* = Inv\langle \overline{X} | \overline{R} \cup R_{HNN} \rangle$, in [3, Theorem 1] it is proved that

$$S^*/\rho \simeq (S_1^{e_1} *_{U^1} S_2^{e_2}) \simeq (S_1 *_U S_2)^1 \tag{1}$$

where $(S_1 *_U S_2)^1$ stands for the free product with amalgamation $S_1 *_U S_2$ with adjoined identity 1, and ρ is the congruence on S^* generated by the relation $t = e_1 = e_2$. Moreover, in [3] it is proved that the Schützenberger automaton of a word $w \in (X \cup X^{-1})^*$ relative to the standard presentation of $S_1 *_U S_2$ may be obtained from the Schützenberger automaton relative to $\langle \overline{X} | \overline{R} \cup R_{HNN} \rangle$ of a special word $w' \in (\overline{X} \cup \overline{X}^{-1})^*$ associate to w. Precisely, consider the following factorization:

$$w = w_1 w_2 \ldots w_{2n-1} w_{2n}$$

where $w_1 \in (X_1 \cup X_1^{-1})^*$, $w_{2i} \in (X_2 \cup X_2^{-1})^+$, $w_{2i+1} \in (X_1 \cup X_1^{-1})^+$, $1 \leq i \leq n-1$ and $w_{2n} \in (X_2 \cup X_2^{-1})^*$. The following associated word

$$w' = w_1 e_1 t e_2 w_2 e_2 t^{-1} e_1 \ldots e_2 t^{-1} e_1 w_{2n-1} e_1 t e_2 w_{2n}$$

is called the *separated normal form* of w. It is significant to recall the following results from [3].

Lemma 1 [3, Lemma 3.4] *Let $w \in (X \cup X^{-1})^+$. The Schützenberger automaton $\mathscr{A}(\overline{X}, R_{HNN} \cup \overline{R} \cup \{t = e_1, t = e_2\}; w)$ may be obtained from the Schützenberger automaton $\mathscr{A}(\overline{X}, R_{HNN} \cup \overline{R}; w')$ of the separated normal form w' of w by identifying the initial and terminal vertices of each t-edge.*

From which it is possible to prove the following proposition.

Proposition 1 [3, Corollary 3.5] *The Schützenberger automaton of a word $w \in (X \cup X^{-1})^+$ relative to the standard presentation $\langle X | R \cup R_W \rangle$ of $S_1 *_U S_2$ may be obtained by deleting all the loops labelled by e_1, e_2 and t in*

$$\mathscr{A}(\overline{X}, R_{HNN} \cup \overline{R} \cup \{t = e_1, t = e_2\}; w).$$

3 The proof of the main result

In this section we prove Theorem 1. The following proposition is a consequence of Lemma 1 and Proposition 1.

Proposition 2 *The Schützenberger automaton of a word $w \in (X \cup X^{-1})^+$ relative to the standard presentation of $S_1^{e_1} *_{U^1} S_2^{e_2}$ of the amalgam $[S_1^{e_1}, S_2^{e_2}; U^1, \omega_1^1, \omega_2^1]$ may be*

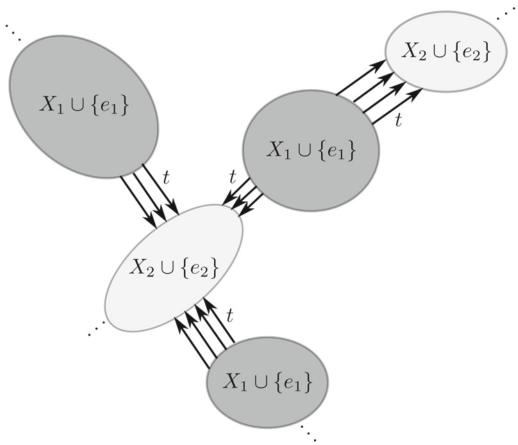

Fig. 1 A graphical representation of a separated t-opuntoid graph

obtained from the Schützenberger automaton $\mathscr{A}(\overline{X}, R_{HNN} \cup \overline{R}; w')$ of the separated normal form w' of w by identifying the initial and terminal vertices of each t-edge and then deleting all the obtained loops labelled by t.

From [3, Lemma 3.3] we may conclude that the automaton $\mathscr{A}(\overline{X}, R_{HNN} \cup \overline{R}; w')$ of Lemma 1 has a particular shape as a weak t-opuntoid automaton on $\overline{X}$ related to the associated HNN-extension. This shape is described in the following definition (see Fig. 1 for a graphical representation).

Definition 4 A weak t-opuntoid automaton (graph) on $\overline{X}$ is called *separated* if the following two conditions hold:

(i) each S-lobe has edges labeled either by elements in $X_1 \cup X_1^{-1} \cup \{e_1, e_1^{-1}\}$ (an S-lobe colored by 1), or $X_2 \cup X_2^{-1} \cup \{e_2, e_2^{-1}\}$ (an S-lobe colored by 2);
(ii) each t-edge is pointing from an S-lobe colored by 1 to a t-adjacent one colored by 2.

We denote by $\mathscr{C}_t$ the set of all separated weak t-opuntoid graphs on $\overline{X}$ related to the associated HNN-extension S^*, and by $\mathscr{C}$ we denote the set of all weak opuntoid graphs on $X \cup \{e_1, e_2\}$ associated to the amalgam $[S_1^{e_1}; S_2^{e_2}, U^1; \omega_1^1, \omega_2^1]$. As it is suggested in Proposition 2 there is a bijection between these two sets. Indeed, we have the following lemma.

Lemma 2 *The map $\psi : \mathscr{C}_t \to \mathscr{C}$ defined by identifying the initial vertex with the terminal one of each t-edge and then erasing the obtained loop, is a bijection.*

Proof Let Γ be a separated weak t-opuntoid graph in $\mathscr{C}_t$. We need to prove that ψ is well defined and the obtained graph $\psi(\Gamma)$ is a weak opuntoid graph (see Fig. 2). Indeed, it is enough to show that in the act of identifying the initial and final vertex of each t-edge there is no pair of S-lobes with the same color that become adjacent in the graph $\psi(\Gamma)$. This follows from the fact that each vertex of a t-edge belongs to exactly one t-edge; this is a consequence of the determinism of Γ and condition (ii) of

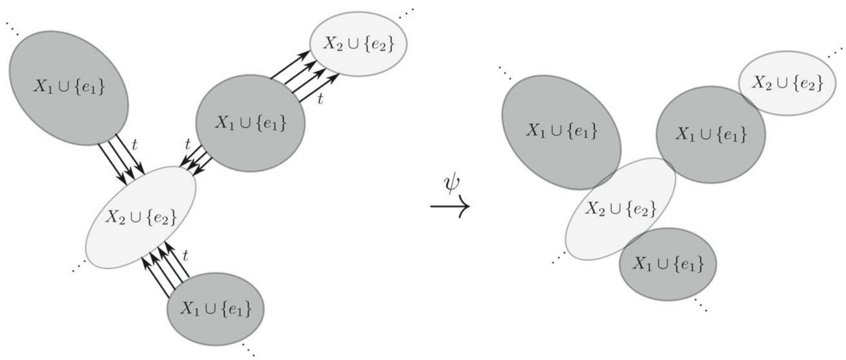

Fig. 2 A graphical representation of the map ψ of Lemma 2

Definition 4. Therefore, there is a one-to-one correspondence between each S-lobe in Γ and the corresponding lobe in $\psi(\Gamma)$, and since no pair of S-lobes are identified in the lobe graph, then the lobe graph of $\psi(\Gamma)$ is a tree. Hence, $\psi(\Gamma)$ is a week opuntoid graph.

The map ψ is a bijection. Indeed, we may define the inverse of map ψ^{-1} by separating adjacent lobes Δ_1, Δ_2 of a weak opuntoid graph Γ' in $\mathscr{C}$ by implanting t-edges between them in the following way. Since Δ_1, Δ_2 share just the intersection vertices $V(\Delta_1) \cap V(\Delta_2)$ we may consider Δ_1, Δ_2 disjoint and so for each $v \in V(\Delta_1) \cap V(\Delta_2)$ we create a new pair of vertices $v(1) \in V(\Delta_1), v(2) \in V(\Delta_2)$ in between which we implant the t-edge $(v(1), t, v(2))$. One may show that this map is well defined and that it is actually the inverse of ψ. □

If, instead of considering the class of weak t-opuntoid (weak opuntoid) graphs, we consider the class of t-opuntoid (weak opuntoid) automata, and we extend the map ψ of Lemma 2 accordingly in the obvious way, then the map ψ is not an isomorphism anymore. The problem is that if α is the initial state and β is the final state of the weak t-opuntoid automaton (α, Γ, β), and α, β belongs to the same t-edges, then in the process of performing the quotient of this edge these two vertices are identified. However, this ambiguity is controlled by property (ii) of Definition 4. Indeed, since a weak t-opuntoid graph is deterministic and by (ii) of Definition 4, each vertex v may have at most one associate t-adjacent vertex $\overline{v}$. It is also evident that these two vertices are identified in an intersection vertex by the map ψ. Thus, we may prove the following lemma.

Lemma 3 *Let $w_1, w_2 \in (X \cup X^{-1})^+$, and let w_1', w_2' be the corresponding separated normal forms of w_1, w_2, respectively. Let $\mathscr{A}(\overline{X}, R_{HNN} \cup \overline{R}; w_1') = (\alpha, \Gamma_1, \beta)$, $\mathscr{A}(\overline{X}, R_{HNN} \cup \overline{R}; w_2') = (\alpha', \Gamma_2, \beta')$ be the corresponding Schützenberger automata that are separated weak t-opuntoid automata such that:*

$$\psi((\alpha, \Gamma_1, \beta)) = \psi\left((\alpha', \Gamma_2, \beta')\right) = (\overline{\alpha}, \overline{\Gamma}, \overline{\beta}).$$

Then, there are $\epsilon_1, \epsilon_2 \in \{0, 1, -1\}$ *such that*

$$t^{\epsilon_1} w_1' t^{\epsilon_2} = w_2' \text{ in } S^*.$$

Proof By Lemma 2 we have that the underlying graphs of $\mathscr{A}(\overline{X}, R_{HNN} \cup \overline{R}; w_1')$ and $\mathscr{A}(\overline{X}, R_{HNN} \cup \overline{R}; w_2')$ are isomorphic. Hence, without loss in generality, we may assume $\Gamma_1 = \Gamma_2 = \Gamma$. Furthermore, by the previous observation $(\alpha', t^{\epsilon_1}, \alpha)$ and $(\beta, t^{\epsilon_2}, \beta')$ are t-edges in Γ for some $\epsilon_1, \epsilon_2 \in \{0, 1, -1\}$, where $\epsilon = 0$ means that the two vertices are equal. Now, using the fact that (α, Γ, β) and $(\alpha', \Gamma, \beta')$ are the Schützenberger automata of w_1', w_2', respectively, and using the language properties of Schützenberger automata [18] we get:

$$w_2' \geq t^{\epsilon_1} t^{-\epsilon_1} w_2' t^{-\epsilon_2} t^{\epsilon_2} \geq t^{\epsilon_1} w_1' t^{\epsilon_2} \geq w_2'$$

from which we derive the statement of the lemma $t^{\epsilon_1} w_1' t^{\epsilon_2} = w_2'$ in S^*. □

Before proving the main result we need three more lemmas. The following one is an easy observation.

Lemma 4 *Let* $w_1, w_2 \in (X \cup X^{-1})^*$, *then* $w_1 = w_2$ *in* $S_1^{e1} *_{U^1} S_2^{e2}$ *if and only if* $w_1 = w_2$ *in* $S_1 *_U S_2$.

Lemma 5 *Let* $w_1, w_2 \in (X \cup X^{-1})^+$, *and let* w_1', w_2' *be the corresponding separated normal forms of* w_1, w_2, *respectively. Then,* $w_1 = w_2$ *in* $S_1 *_U S_2$ *if and only if there are* $\epsilon_1, \epsilon_2 \in \{0, 1, -1\}$ *such that*

$$t^{\epsilon_1} w_1' t^{\epsilon_2} = w_2' \text{in } S^*.$$

Proof The "if part" is a consequence of (1) and Lemma 4. To prove the "only if" part let $\mathscr{A}(\overline{X}, R_{HNN} \cup \overline{R}; w_1') = (\alpha, \Gamma_1, \beta)$, $\mathscr{A}(\overline{X}, R_{HNN} \cup \overline{R}; w_2') = (\alpha', \Gamma_2, \beta')$ be the corresponding Schützenberger automata. Since $w_1 = w_2$ in $S_1 *_U S_2$, then by Lemma 4 we have $w_1 = w_2$ in $S_1^{e1} *_{U^1} S_2^{e2}$. Thus, by Proposition 2, and the definition of the morphism ψ we get

$$\psi\left((\alpha, \Gamma_1, \beta)\right) = \psi\left((\alpha', \Gamma_2, \beta')\right).$$

Hence, the statement follows by Lemma 3. □

The last lemma regards the finiteness of the $\mathscr{R}$-classes in the free product of inverse semigroups, for completeness we report here in form of lemma.

Lemma 6 [11, Proposition 5.1] *If* S_1, S_2 *are semigroups with finite* $\mathscr{R}$-*classes, then the free product* $S_1 * S_2$ *has finite* $\mathscr{R}$-*classes.*

We are now in position to prove the main theorem of the paper.

Proof (of Theorem 1) Assume, contrary to the statement, that the word problem in an HNN-extension with the conditions stated in the theorem is decidable. Consider any amalgam of inverse semigroups $[S_1, S_2; U, \omega_1, \omega_2]$ satisfying the conditions of Theorem 2 which are:

- S_1, S_2 have finite $\mathscr{R}$-classes;
- U is a free inverse semigroup with zero of finite rank;
- the membership problem of $\omega_i(U)$ is decidable in S_i for $i = 1, 2$;
- ω_1, ω_2 and their inverses are computable functions;

and consider the associated Yamamura's HNN-extension:

$$S^* = [S_1^{e_1} * S_2^{e_2}; U_1^{e_1}, U_2^{e_2}; (\omega_1^1)^{-1} \circ \omega_2^1]. \tag{2}$$

By Lemma 6 we have that $S_1^{e_1} * S_2^{e_2}$ has finite $\mathscr{R}$-classes. Furthermore, $U_1^{e_1} \simeq U_2^{e_2}$ is a free inverse monoid with zero of finite rank, and both $(\omega_1^1)^{-1} \circ \omega_2^1$ and $(\omega_2^1)^{-1} \circ \omega_1^1$ are computable functions. Since the membership problem of $\omega_i(U)$ is decidable in S_i for $i = 1, 2$, then the same occurs for $U_1^{e_1}, U_2^{e_2}$ in $S_1^{e_1} * S_2^{e_2}$. Therefore, S^* of (2) is a Yamamura's HNN-extensions satisfying the conditions of the statement, and by our assumptions it has solvable word problem. However, if we put $S_1 *_U S_2 = Inv\langle X|R \cup R_w\rangle$ by Lemma 5 we may decide whether or not two words $w_1, w_2 \in (X \cup X^{-1})^*$ are equal in $S_1 *_U S_2$ by simply building (effectively) the associated separated normal forms w_1', w_2', and then using the decidability of the word problem for the HNN-extension (2) to effectively test whether or not there are $\epsilon_1, \epsilon_2 \in \{0, 1, -1\}$ such that

$$t^{\epsilon_1} w_1' t^{\epsilon_2} = w_2' \text{ in } S^*.$$

Since there are finitely many cases to consider, this is a decidable task, hence the word problem for $S_1 *_U S_2$ is decidable. However, this contradicts Theorem 2. □

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References

1. Ash, C.: Embedding theorem using amalgamation bases. In: Hall, T.E., Jones, P.R., Preston, G.B. (eds.) Semigroups, pp. 167–176. Academic Press, New York (1980)
2. Cherubini, A., Meakin, J.C., Piochi, B.: Amalgams of finite inverse semigroups. J. Algebra **285**, 706–725 (2005)
3. Cherubini, A., Rodaro, E.: Amalgams vs Yamamura's HNN-extensions of inverse semigroups. Algebra Colloq. **18**(04), 647–657 (2011)
4. Dekov, D.V.: HNN-extensions of inverse semigroups. Semigroup Forum **49**, 83–87 (1994)
5. Gilbert, N.D.: HNN-extensions of inverse semigroups and grupoid. J. Algebra **272**, 27–45 (2004)
6. Hall, T.E.: Inverse and regular semigroups and amalgamations: a brief survey. Proc. Sympos. Semigroups, Nothern Illinois Univ., De Kalb **I11**, 49–79 (1979)
7. Howie, J.M.: Embedding theorems for semigroups. Q. J. Math. **2**(14), 254–258 (1963)
8. Howie, J.M.: Automata and Languages. Clarendon Press, Oxford (1991)
9. Jackson, D.: A normal form theorem for Higman-Neumann-Neumann extensions of semigroups. Ph.D. Thesis, University of Illinois (1978)
10. Jajcayova, T.: HNN-extensions of inverse semigroups. PhD Thesis at University of Nebraska-Lincoln Department of Mathematics and Statistics (1997)

11. Jones, P.R., Margolis, S.W., Meakin, J.C., Stephen, J.B.: Free products of inverse semigroups II. Glasgow Math. J. **33**(3), 373–387 (1991)
12. Lawson, M.: Inverse Semigroups. World Scientific Publishing Co. Pte. Ltd, Singapore (1998)
13. Lyndon, R. C., Schupp, P. E.: Combinatorial Group Theory. "Classics in Mathematics" series, reprint of the 1977 edition. World Scientific Publishing Co. Pte. Ltd, Singapore (2001)
14. Munn, W.D.: Free inverse semigroups. Proc. Lond. Math. Soc. **29**(3), 385–404 (1974)
15. Petrich, M.: Inverse Semigroups. Wiley, New York (1984)
16. Rodaro, E., Cherubini, A.: Decidability of the word problem in Yamamura's HNN-extensions of finite inverse semigroups. Semigroup Forum **77**(2), 163–186 (2008)
17. Rodaro, E., Silva, P.V.: Amalgam of inverse semigroups and reversible two-counter machines. J. Pure Appl. Algebra **217**(4), 585–597 (2013)
18. Stephen, J.B.: Presentation of inverse monoids. J. Pure Appl. Algebra **198**, 81–112 (1990)
19. Yamamura, A.: HNN-extensions of inverse semigroups and applications. Int. J. Algebra Comput. **7**(5), 605–624 (1997)
20. Yamamura, A.: Locally full hnn extensions of inverse semigroups. J. Austral. Math. Soc. **70**, 235–272 (2001)
21. Yamamura, A.: Embedding theorems for HNN-extensions of inverse semigroups. J. Pure Appl. Algebra **210**, 521–535 (2007)